3.24.

Nature de $\sum_{k\geq 1} \frac{\cos(k)}{k}$.

On pose $\begin{cases} \forall n \in \mathbb{N}^*, \ S_n = \sum_{k=1}^{n} \cos(k) \\ S_0 = 0 \end{cases}$.

Soit $n \in \mathbb{N}^*$.

$$\sum_{k=1}^{n} \frac{\cos(k)}{k} = \sum_{k=1}^{n} \frac{S_k - S_{k-1}}{k} = \sum_{k=1}^{n} \frac{S_k}{k} - \sum_{k=1}^{n} \frac{S_{k-1}}{k}.$$

$$\sum_{k=1}^{n} \frac{\cos(k)}{k} = \sum_{k=1}^{n} \frac{S_k}{k} - \sum_{k=0}^{n-1} \frac{S_{k-1}}{k+1} = \sum_{k=1}^{n-1} \frac{S_k}{k} - \sum_{k=1}^{n-1} \frac{S_k}{k+1} + \frac{S_n}{n}.$$

$$\sum_{k=1}^{n} \frac{\cos(k)}{k} = \sum_{k=1}^{n-1} S_k \left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{S_n}{n}$$

$$S_n = \mathrm{Re}\left(\sum_{k=1}^{n} e^{ik}\right) = \mathrm{Re}\left(e^{i} \frac{1-e^{in}}{1-e^{i}}\right)$$

donc

$$|S_n| \leq \left| e^{i} \frac{1-e^{in}}{1-e^{i}} \right| \leq \frac{2}{|1-e^{i}|} \leq \frac{2}{|2\sin(\frac{1}{2})|} \leq \frac{1}{\sin(\frac{1}{2})}.$$

$(S_n)_{n\in\mathbb{N}}$ est donc bornée puis $\frac{S_n}{n} \xrightarrow[n\to+\infty]{} 0$.

$\forall k \in \mathbb{N}^*, \ \left|S_n\left(\frac{1}{k} - \frac{1}{k+1}\right)\right| \leq \frac{1}{\sin(\frac{1}{2})}\left(\frac{1}{k} - \frac{1}{k+1}\right)$ donc $\sum_{k\geq 1} S_n\left(\frac{1}{k} - \frac{1}{k+1}\right)$ CVA.

donc $\sum_{k\geq 1} S_n\left(\frac{1}{k} - \frac{1}{k+1}\right) \leq \int_n^{n+1} e^{-t} f(t)\,dt \leq f(n) \int_n^{n+1} e^{-t}\,dt$

$\sum_{n\geq 1} \frac{\cos(n)}{n}$ n'est pas ACV

$$\forall n \in \mathbb{N}^*, \ \frac{|\cos(n)|}{n} \geq \underbrace{\frac{\cos^2(n)}{n}}_{\frac{1}{n}\frac{1+\cos(2n)}{2}}.$$

Or, $\sum_{n\geq 1} \frac{\cos(2n)}{n}$ CV et $\sum_{n\geq 1} \frac{1}{n}$ DV donc $\sum_{n\geq 1} \frac{|\cos(n)|}{n}$ DV.